# THE PHELPS-KOOPMANS THEOREM AND POTENTIAL OPTIMALITY

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## 1. Introduction

In the aggregative growth model, the Phelps-Koopmans theorem provides one of the most well-known sufficient conditions for inefficiency. Awarding the 2006 Nobel Prize in Economic Sciences to Edmund Phelps, the Royal Swedish Academy of Sciences observed that

> "Phelps . . . showed that all generations may, under certain conditions, gain from changes in the savings rate."

These "certain conditions" were conjectured by Phelps (1962), and the verification of these conditions, based on a proof provided by Koopmans, appeared in Phelps (1965). The Phelps-Koopmans theorem may be stated as follows: if every limit point of a path of capital stocks exceeds the "golden rule", then that path is inefficient: there is another feasible path from the same initial stock which provides at least as much consumption at every date and strictly more consumption at some date.

Why might such a result be noteworthy? While efficiency is of intrinsic interest, one might argue that the real interest is in *optimal* programs, in which a "sum" of one-period utilities is maximized. When there is discounting, that sum is indeed well-defined, and it was already well-known that optimal programs must converge to a "modified" golden rule, which lies below the golden rule. In the light of this result, what does the Phelps-Koopmans theorem add?

Tapan Mitra, a leading theorist who has made deep contributions to our understanding of the aggregative growth model, has this to say:

> "Both Phelps and Koopmans would have known [about convergence to the modified golden rule] in 1965; yet they were interested in the Phelps-Koopmans theorem. Since Phelps explicitly brings in the optimality notion, the only way this makes sense to me is that he had in mind the optimal growth model without discounting, or at least he had not completely made up his mind about the discounting issue. Koopmans (1960) provided what is considered to be the definitive axiomatic treatment of discounted utilitarianism, but he was quite uncomfortable with the result, as his writings in 1964 and 1972 on this issue clearly show.
>
> [The exercise] can only be justified, from the optimality point of view, if one is willing to allow undiscounted optimality. Inefficiency is, after all, about

[1]The aggregative growth model represented my point of entry into research in 1979, and Tapan Mitra was the one who opened that door. It's only fitting, then, that I should dedicate this paper, with affection and gratitude, to Tapanda on the occasion of his 60th birthday. May there be many more occasions to celebrate the work of this remarkable theorist!

> spending an infinite amount of time above the golden rule. It has no interest for discounted optimization theory."[2]

In the above quote Mitra refers, of course, to the *convex* model of growth, for which the Phelps-Koopmans theorem is stated, and for which the discounted optimality result was well-known. The discomfort that he alludes to would have been entirely justified, given the philosophical objections to discounting raised in the early work of Ramsey (1927). Yet it does raise the question of whether the same observations are true in a nonconvex model of growth: can *discounted* optimal paths spend an "infinite amount of time above" the golden rule?

This paper seeks to answer that question. I show in Proposition 1 that if an optimal path converges, its limit must lie weakly below the *minimal golden rule*, the lowest capital stock that globally maximizes net consumption. This result is independent of any curvature assumptions, either on the production function or on the utility function. Thus far, then, the intuition of the convex model carries over: convergent programs that are potentially optimal with respect to some utility function cannot stay above and bounded away from the golden rule. As an aside, it is of some interest that the same observation is not generally true of *efficient* programs: Mitra and Ray (2008) show by example that an efficient convergent path might converge to a limit that strictly exceeds the minimal golden rule.

However, the main result of the paper is Proposition 3. I prove that there are potentially optimal paths *with discounting* which have the property that they perennially lie above (and bounded away from) the minimal golden rule. Indeed, I construct such paths in a framework with a *unique* golden rule. So the word "minimal" can safely be dropped from the statement of the proposition.[3]

## 2. Preliminaries

At every date, capital $k_t$ produces output $f(k_t)$, where $f : \mathbb{R} \to \mathbb{R}$ is the production function. We assume throughout that $f$ satisfies the following restrictions:

[F.1] $f$ is increasing and continuous, with $f(0) = 0$.

[F.2] There is $K \in (0, \infty)$ such that $f(x) > x$ for all $x \in (0, K)$ and $f(x) < x$ for all $x > K$.

We refer to $K$ as the *maximum sustainable stock*. Observe that $f$ is permitted to be nonconcave.

A *feasible path* from $\kappa \geq 0$ is a sequence of *capital stocks* $\mathbf{k} \equiv \{k_t\}$ with

$$k_0 = \kappa \text{ and } 0 \leq k_{t+1} \leq f(k_t)$$

for all $t \geq 0$. Associated with the feasible path $\mathbf{k}$ from $\kappa$ is a *consumption sequence* $\{c_t\}$, defined by

$$c_t = f(k_{t-1}) - k_t \text{ for } t \geq 1.$$

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[2]This quote is extracted from a private communication by Tapan Mitra.

[3]Mitra and Ray (2008) show, again by example, that there is an efficient path which lies above and bounded away from a unique golden rule. Proposition 3 represents a significant strengthening of this result.

It is obvious that for every feasible path $\mathbf{k}$ from $\kappa$, both $k_t$ and $c_{t+1}$ are bounded above by $\max\{K, \kappa\}$. With no real loss of generality, we presume that $\kappa \in [0, K]$, so that for every feasible path $\mathbf{k}$ from $\kappa$,

$$k_t \leq K \text{ for } t \geq 0 \text{ and } c_t \leq K \text{ for } t \geq 1.$$

A feasible path $\mathbf{k}$ from $\kappa$ is *inefficient* if there is a feasible path $\mathbf{k}'$ from $\kappa$ such that

$$c'_t \geq c_t \text{ for all } t \geq 1,$$

with strict inequality for some $t$. It is *efficient* if it is not inefficient.

Under [F.1] and [F.2] there is $z \in (0, K)$ such that

$$f(z) - z \geq f(x) - x \text{ for all } x \geq 0.$$

Then we call $z$ a *golden rule stock*, or simply a *golden rule*. Certainly, there can be several golden rules, all in $(0, K)$. But a minimal golden rule — the smallest of all the golden rule stocks — must exist, which we denote by $\gamma$. *Golden rule consumption* is, of course, the same for all golden rules; it is denoted by $c$.

For any $\delta \in (0, 1)$, a feasible path $\mathbf{k}^*$ from $\kappa$ is *potentially $\delta$-optimal* if there exists a strictly increasing and continuous utility function $u$ defined on consumption such that $\mathbf{k}^*$ solves the problem

$$\max_{\mathbf{k}} \sum_{t=0}^{\infty} \delta^t u(c_{t+1})$$

where $\{c_t\}$ is the consumption path associated with $\mathbf{k}$. The path is *potentially optimal* if it is potentially $\delta$-optimal for some $\delta \in (0, 1)$.

The following observation is trivial.

Observation **1.** *If a feasible path is potentially optimal, then it is efficient.*

*Remark.* The converse is not true; see below.

## 3. Potential Optimality for Convergent Paths

The following result shows that all potentially optimal *convergent* paths must have a limit that lies below the minimal golden rule.

Proposition **1.** *Suppose that $\mathbf{k}^*$ is a potentially optimal path with a well-defined limit $k^*$. Then $k^* \leq \gamma$.*

*Proof.* Suppose that $\mathbf{k}^*$ has limit $k^* > \gamma$. Define $c^* \equiv f(k^*) - k^*$. Then associated consumption $c^*_t \to c^*$. Fix any strictly increasing and continuous utility function $u$. Then there exists $\epsilon > 0$ and $T' \geq 1$ such that for all $t \geq T'$,

$$u\left(c^*_t + k_t - \gamma\right) - u\left(c^*_t\right) \geq 2\epsilon. \tag{1}$$

Choose $T \geq T'$ such that

$$\sum_{t=T+1}^{\infty} \delta^{t-T} \left[u(c_t) - u(c^*)\right] < \epsilon, \tag{2}$$

Define a new path $\mathbf{k}$ from the same initial stock by $k_t = k_t^*$ for all $t \leq T-1$, and $k_t = \gamma$ for all $t \geq T$. By (1), $k_T^* > \gamma$, so $\mathbf{k}$ is feasible. Let $c \equiv f(\gamma) - \gamma$ stand for golden rule consumption. Note that $c \geq c^*$. Then the overall payoff generated by $\mathbf{k}$ is

$$\begin{aligned} v(\mathbf{k}) &\equiv \sum_{t=1}^{T-1} \delta^{t-1} u(c_t) + \delta^{T-1} u\left(c_T + k^* - \gamma\right) + \sum_{t=T+1}^{\infty} \delta^{t-1} u(c) \\ &\geq \sum_{t=1}^{T-1} \delta^{t-1} u(c_t) + \delta^{T-1} u\left(c_t^*\right) + 2\delta^{T-1}\epsilon + \sum_{t=T+1}^{\infty} \delta^{t-1} u(c^*) \\ &\geq \sum_{t=1}^{T-1} \delta^{t-1} u(c_t) + \delta^{T-1} u\left(c_t^*\right) + 2\delta^{T-1}\epsilon + \sum_{t=T+1}^{\infty} \delta^{t-1} u(c_t) - \delta^{T-1}\epsilon \\ &= \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) + 2\delta^{T-1}\epsilon \\ &> v(\mathbf{k}^*), \end{aligned}$$

where the first inequality uses (1) and $c \geq c^*$, and the second inequality uses (2).

This proves that $\mathbf{k}^*$ cannot be potentially optimal. ∎

*Remarks*

1. It is obvious that a potentially optimal program is efficient. This proposition, combined with Example 1 in Mitra and Ray (2008), shows that the converse is generally false. That example constructs a convergent efficient program with limit capital stock above the minimal golden rule. By Proposition 1, such a program cannot be potentially optimal.

2. One might argue that convergent paths are unable to exploit consumption gains from a nonconvex technology by cycling through various capital stocks. (And this is why the "traditional" result of Proposition 1 is to be had for such paths.) But this intuition is erroneous. To the right of the minimal golden rule no such gain exists. To see this informally, suppose that a path $\mathbf{k}$ repeatedly cycles through the $n$ values $k^1, \ldots, k^n$. Define $c^1 \equiv f(k^i) - k^{i+1}$, where the addition $i+1$ is modulo $n$. Can the average consumption

$$\frac{1}{n}[c^1 + \cdots + c^n]$$

ever exceed golden rule consumption $c$? It cannot. To see why, simply note that

$$\frac{1}{n}[c^1 + \cdots + c^n] = \frac{1}{n}\sum_{i=1}^{n}[f(k^i) - k^{i+1}] = \frac{1}{n}\sum_{i=1}^{n}[f(k^i) - k^i]] \leq c.$$

Indeed, the following proposition drives this point home:

Proposition 2. *If* $\mathbf{k}$ *is an optimal path under some increasing, continuous and strictly concave* $u$*, then it must converge.*

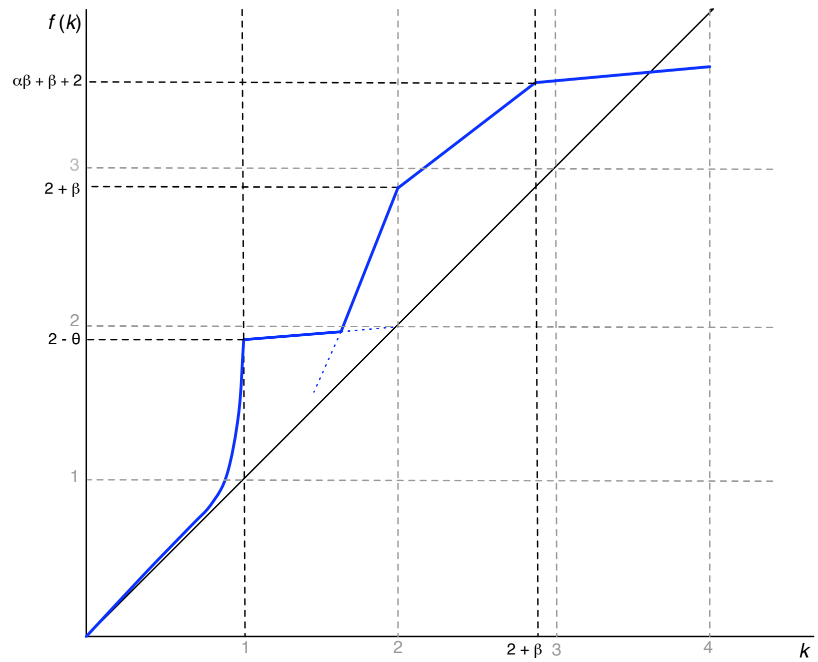


FIGURE 1. THE FUNCTION $f^{\epsilon}$ DEFINED IN EQUATION (4)

The proof may be found in Mitra and Ray (1984) and is omitted here. The proposition underscores the observation that no matter how small the degree of concavity in $u$, it never pays to cycle (nor engage in more exotic nonconvergent behavior).

## 4. THE PHELPS-KOOPMANS THEOREM AND POTENTIAL OPTIMALITY

Proposition 1 is well-known for concave models. This is why the study of paths that stay above the golden rule for an infinite number of periods is typically not justified by the optimal growth model *with discounting*. The purpose of this section is to show that this observation fails for nonconvex models.

I require throughout that $f$ satisfy [F.1] and [F.2]. I impose an additional restriction in a deliberate attempt to stay close in spirit to the model with convex technology:

[F.3] $f$ has a unique golden rule $\gamma$.

(Note that additional requirements on $f$ make our result below stronger, not weaker.)

PROPOSITION **3.** *For every $\delta$ such that $\delta^2 + \delta^3 > 1$, there exists a technology satisfying* [F.1]–[F.3], *an initial stock $\kappa > 0$ and a potentially $\delta$-optimal path $\mathbf{k}^*$ from $\kappa$, with $\inf_t k_t^* > \gamma$.*

*Proof.* We begin by constructing a technology. Fix $\alpha \in (0.9, 1)$, $\beta \in (0.9, 1)$ and $\theta \in (0, 1 - \beta)$. Define

$$\epsilon_0 \equiv 1 - \beta. \tag{3}$$

For any $\epsilon \in (0, \epsilon_0)$, define

$$f^{\epsilon}(k) = \begin{cases} k + (1-\theta)k^{1/\epsilon} & \text{for } 0 \le k < 1 \\ \max\left\{\theta(k-1) + (2-\theta), (2+\beta) - \frac{\beta}{\epsilon}(2-k)\right\} & \text{for } 1 \le k < 2 \\ \alpha(k-2) + (2+\beta) & \text{for } 2 \le k < 2+\beta \\ (\alpha\beta + \beta + 2) + \epsilon\left[k - (2+\beta)\right] & \text{for } k \ge 2+\beta. \end{cases} \tag{4}$$

It is easy to see that $f^{\epsilon}$ satisfies [F.1] and [F.2] for all $\epsilon \in (0, \epsilon_0)$. Figure 1 illustrates $f^{\epsilon}$. In particular, for $\epsilon < \epsilon_0$, the two affine segments between 1 and 2 do indeed intersect in the way shown in the diagram.

[F.3] is also satisfied. The only two candidates for a golden rule are $k = 1$ and $k = 2$ (see Figure 1 again). Because we've assumed that $\theta < 1 - \beta$, we see that $f^{\epsilon}(1) - 1 = 1 - \theta > f^{\epsilon}(2) - 2 = \beta$.

Denote by $K$ the maximum sustainable capital stock under $f$. These increase with $\epsilon$. We will denote the largest of these, $K^{\epsilon_0}$, by $M$. No one-period value of output, capital or consumption over $c^*$ can exceed this amount, given that we shall always consider $\kappa \in (0, M)$.

Consider the initial stock $\kappa^* = 2 + \beta$. Define the path $\mathbf{k}^*$ given by $k_t = 2 + \beta$ for $t$ even, $k_t = 2$ for $t$ odd. It is feasible (independently of the specific value of $\epsilon \in (0, \epsilon_0)$). The consumption sequence associated with this path is given by $c_t^* = c^* \equiv (\alpha + 1)\beta$ for $t$ even, and $c_t^* = 0$ for $t$ odd. Observe that $\inf_t k_t^* > \gamma$.

We are going to show that $\mathbf{k}^*$ is potentially $\delta$-optimal.

To this end, pick $0 < \epsilon_1 \le \epsilon_0$ such that for all $\epsilon \in (0, e_1)$,

$$2\epsilon < -\ln\delta. \tag{5}$$

For $\epsilon \in (0, \epsilon_1)$, define

$$u^{\epsilon}(c) = \begin{cases} 0 & \text{for } c = 0 \\ \delta^{(c^*-c)/\epsilon c} & \text{for } 0 < c \le c^* \\ 1 + \delta^{(M-c)/(\epsilon[c-c^*])} & \text{for } c > c^*. \end{cases} \tag{6}$$

Claim 1. *Provided that $\epsilon \in (0, \epsilon_1)$, $u^{\epsilon}$ is strictly increasing and continuous, and $u^{\epsilon\prime\prime}(c) > 0$ for all $c \in (0, M]$ with $c \neq c^*$.*

*Proof.* It is trivial to verify that $u^{\epsilon}$ is increasing and continuous. Differentiate $u^{\epsilon}$ at any $c \in (0, c^*)$:

$$u^{\epsilon\prime}(c) = -\frac{\delta^{(c^*-c)/\epsilon c} c^* \ln\delta}{c^2\epsilon},$$

and differentiate again to see that

$$u^{\epsilon\prime\prime}(c) = \frac{\delta^{(c^*-c)/\epsilon c} c^* \ln\delta}{c^4\epsilon}\left[(\ln\delta)\frac{c^*}{\epsilon} + 2c\right].$$

Because $\ln\delta < 0$, we see that $u^{\epsilon\prime\prime}(c) > 0$ if

$$(\ln\delta)\frac{c^*}{\epsilon} + 2c < 0$$

but this inequality is assured by (5).

A similar argument verifies that $u^{\epsilon\prime\prime}(c) > 0$ for any $c \in (c^*, M]$. ∎

Our main argument will rely on showing that paths *other* than $\mathbf{k}^*$ are *not* optimal. Such an argument relies on the presumption that an optimal program always exists, which — given the continuity of $u^\epsilon$, along with $\delta \in (0,1)$ as well as [F.1] and [F.2] — is an entirely standard proposition.

For any initial stock $\kappa$, denote by $V^\epsilon(\kappa)$ the optimal value of starting from $\kappa$ (under the utility function $u^\epsilon$). The following claim follows trivially from an inspection of (4) and (6): there is a function $h(\epsilon)$ with $h(\epsilon) \to 0$ as $\epsilon \to 0$ such that

$$V^\epsilon(2+\beta) \geq V^\epsilon(K^\epsilon) - h(\epsilon), \tag{7}$$

where $K^\epsilon$, we recall, is the maximum sustainable capital stock under $f$.

What follows is a list of additional restrictions on $\epsilon$. For the purposes of readability, it may be useful to skip ahead to the argument following restriction (17), referring back to these conditions when needed. First, select $\eta$ so that

$$0 < \eta < \alpha(1-\beta). \tag{8}$$

Next, define

$$\nu \equiv \eta/4. \tag{9}$$

Now find a threshold $\epsilon_2 \in (0, \epsilon_1)$ such that for all $\epsilon \in (0, \epsilon_2)$,

$$\frac{1}{1+\delta} > u^\epsilon(c^* - \eta) + \delta h(\epsilon) + \delta^{\nu/\epsilon}, \tag{10}$$

where $h(\epsilon)$ is the function in (7). Such a threshold can obviously be found because all terms on the right-hand side of (10) converge to 0 as $\epsilon \to 0$.

Next, choose a positive integer $N$ and a threshold $\epsilon_3 \in (0, \epsilon_2)$ such that for all $0 < \epsilon < \epsilon_3$,

$$\frac{1}{1-\delta^2} > (1+\delta+\delta^2)u^\epsilon(M) + (\delta^3 + \cdots + \delta^{N+2})u^\epsilon(c^* - \eta) + \frac{\delta^{N+3}u^\epsilon(M)}{1-\delta} + \delta^{\nu/\epsilon}. \tag{11}$$

To see why such thresholds must exist, observe that $u^\epsilon(M) \to 1$, $\delta^{\nu/\epsilon} \to 0$ and $u^\epsilon(c^* - \eta) \to 0$ as $\epsilon \to 0$, and $\delta^N \to 0$ as $N \to \infty$. Consequently, the required thresholds are available provided that

$$\frac{1}{1-\delta^2} > 1 + \delta + \delta^2,$$

but this inequality follows from the given restriction on $\delta$ in the statement of the proposition.

Choose $\epsilon_4 \in (0, \epsilon_3)$ such that

$$f^{\epsilon(N)}(1/2) < c^* - \eta \tag{12}$$

for all $\epsilon \in (0, \epsilon_4)$, where the superscript $(N)$ denotes $N$-fold composition, and $N$ has been chosen above to satisfy (11). Such a threshold $\epsilon_2$ is available because $f^{\epsilon(N)}(1/2) \to 1/2$ as $\epsilon \to 0$, and because $c^* - \eta = (\alpha+1)\beta - \eta > 3/2$, by (8).[4]

Using (4), the inequality (12) may be interpreted as follows: starting from an initial stock of $1/2$, it will require more than $N$ periods of pure accumulation to bring total output up to $c^* - \eta$.

For our next restriction, it will be useful to define $\zeta(\epsilon)$ to be the intersection of the two affine segments that define $f^\epsilon$ on the interval $[1,2]$ (see (4) and Figure 1). Formally, $\zeta(\epsilon)$ is the solution in $k$ to

$$\theta(k-1) + (2-\theta) = (2+\beta) - \frac{\beta}{\epsilon}(2-k). \tag{13}$$

It is easy to see that $\zeta(\epsilon) \in (1,2)$ as long as $\epsilon < 1 - \beta$ (which we've assumed already), and that $\zeta(\epsilon) \to 2$ as $\epsilon \to 0$. For $x > 1$, define $\phi(x) \equiv \theta(k-1) + (2-\theta)$. Choose $\epsilon_5 \in (0, \epsilon_4)$ such that

$$\phi_\epsilon^{(N)}(3/2) < \zeta(\epsilon) \tag{14}$$

for all $\epsilon \in (0, \epsilon_5)$. This threshold is available because $\zeta(\epsilon) \to 2$ as $\epsilon \to 0$, while $\phi_\epsilon^{(N)}(3/2) < 2$.

Using (4), the inequality (14) may be interpreted as follows: starting from an initial stock of $3/2$, even $N$ periods of "pure accumulation" will keep total output below the value of 2.

Next, pick $\epsilon_6 \in (0, \epsilon_5)$ such that for all $\epsilon \in (0, \epsilon_6)$,

$$3\epsilon + \frac{2\epsilon^2}{\alpha\beta - \epsilon} < \nu. \tag{15}$$

Our final restriction on $\epsilon$ will require some preparatory work. For $k \in [0, K]$, define $\sigma(k) \equiv f^\epsilon(f^\epsilon(k)) - k$.

Claim **2.** *$\sigma(2) > \sigma(k)$ for every $k \neq 2$.*

*Proof.* It is very easy to see, given our restrictions, that there are only two potential maximizers of $\sigma(k)$, and they are $k = 1$ and $k = 2$. Direct computation shows that $\sigma(2) = (\alpha+1)\beta = c^*$. To compute $\sigma(1)$, note that

$$f^\epsilon(1) = 2 - \theta < f^\epsilon(3/2) \leq \phi(3/2) \leq \phi^{(N)}(3/2) < \zeta(\epsilon),$$

where the last inequality follows from (14). Therefore $f^\epsilon(f^\epsilon(1)) < 2$. It follows that $\sigma(1) < 1$, but this is smaller than $c^* = \sigma(2)$. Therefore $\sigma(2) > \sigma(k)$ for all $k \neq 2$. ∎

Define $\xi(\epsilon)$ by $f^\epsilon(\xi(\epsilon)) \equiv 2$. It is easy to see that

$$\zeta(\epsilon) < \xi(\epsilon) < 2, \tag{16}$$

where $\zeta(\epsilon)$ is defined by the solution to (13).[5]

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[4]The last observation follows from the fact that $(\alpha+1)\beta - \eta > (\alpha+1)\beta - \alpha(1-\beta) = 2\alpha\beta + \beta - \alpha$. Given that $\alpha, \beta \in (0.9, 1)$, this exceeds $3/2$.

[5]Because $f^\epsilon(2) > 2$, $\xi(\epsilon) < 2$. Because $f^\epsilon(\zeta(\epsilon)) = \theta[\zeta(\epsilon) - 1] + (2-\theta) < 2$, we have $\xi(\epsilon) > \zeta(\epsilon)$.

Define $S$ to be the maximum value of $\sigma(k)$ on $[0, \xi(\epsilon)]$. By Claim 2, we know that

$$(\alpha+1)\beta = \sigma(2) > S.$$

Pick $\epsilon_7 \in (0, \epsilon_6)$ such that for all $\epsilon \in (0, \epsilon_7)$,

$$\epsilon\left[\frac{4}{(\alpha+1)\beta - S} + 3\right] < \nu. \tag{17}$$

This completes all our restrictions on $\epsilon$. In the remainder of the proof, I *fix* some value of $\epsilon \in (0, \epsilon_7)$. To emphasize that $\epsilon$ will remain unchanged for the rest of the argument (and to simplify notation), I will refer to $u^\epsilon$ as $u$ and $f^\epsilon$ as $f$, and omit similar superscripts from other relevant objects (such as value functions or the maximum sustainable capital stock).

I will prove that $\mathbf{k}^*$ is optimal under the utility function $u$. I proceed in several steps.

We consider below various feasible paths $\mathbf{k}$, from $\kappa^*$ as well from other initial stocks. Denote by $V_t$ the corresponding values generated by $\mathbf{k}$ at each date $t \geq 1$. The main value of interest is, of course, $V_1$, which we denote by $v(\mathbf{k})$.

Claim 3. *Let $\mathbf{k}$ be a feasible path from some initial stock in $[0, K]$, and suppose that associated consumption on this path satisfies $c_1 < c^* - \eta$. Then $v(\mathbf{k}) < V(\kappa^*) - \delta^{\nu/\epsilon}$.*

*In particular, no such path can be optimal from $\kappa^*$.*

*Proof.* Associated consumption on the path $\mathbf{k}$ satisfies $c_1 < c^* - \eta$. Therefore

$$\begin{aligned}
v(\mathbf{k}) &< u(c^* - \eta) + \delta V_2 \\
&\leq u(c^* - \eta) + \delta V(K) \\
&\leq u(c^* - \eta) + \delta h(\epsilon) + \delta V(\kappa^*) \\
&= u(c^* - \eta) + \delta h(\epsilon) - (1-\delta)V(\kappa^*) + V(\kappa^*) \\
&\leq u(c^* - \eta) + \delta h(\epsilon) - \frac{1}{1+\delta} + V(\kappa^*) \\
&< V(\kappa^*) - \delta^{\nu/\epsilon},
\end{aligned}$$

where the third inequality uses (7), the penultimate inequality employs the fact that $V(\kappa^*) \geq 1/(1-\delta^2)$, which is the value generated by $\mathbf{k}^*$ from $\kappa^*$, and the very last inequality invokes (10). ∎

Claim 4. *Let $\mathbf{k}$ be a feasible path from some initial stock in $[0, K]$, and suppose that associated consumption on this path satisfies $c_t \geq c^* - \eta$ for $t = 1, 2$. Then*

$$k_2 < \frac{3}{2}.$$

*Proof.* Because $\epsilon < 1 - \beta$ (see (10), note first that $K \leq M$, where $M$ is the maximum sustainable capital stock when $\epsilon = 1 - \beta$. It is easy to see that

$$M = \alpha + \beta + 2, \tag{18}$$

so that given $c_1 \geq c^* - \eta$, it must be that $k_1 \leq (2+\eta) + \alpha(1-\beta)$. So

$$f(k_1) \leq f\left([2+\eta] + \alpha[1-\beta]\right) \leq \alpha[\eta + \alpha(1-\beta)] + (2+\beta),$$

so that (remembering $c_2 \geq c^* - \eta$)

$$k_2 \leq (1+\alpha)\eta + \alpha^2(1-\beta) + (2-\alpha\beta) < \frac{3}{2},$$

where the second inequality uses (8).[6] ∎

Claim 5. *Let* $\mathbf{k}$ *be a feasible path from some initial stock in* $[0, K]$, *and suppose that* $k_t < 3/2$ *for some* $t = 0, 1, 2$. *Then* $v(\mathbf{k}) \leq V(\kappa^*) - \delta^{\nu/\epsilon}$.

*Proof.* Suppose that $k_t < 3/2$ for some $t \leq 2$. Now recall the two remarks of interpretation following (12) and (14). They jointly imply that at most *one* round of consumption $c^* - \eta$ or better can be sustained over the next $N+1$ periods. It follows that

$$\begin{aligned} v(\mathbf{k}) &\leq (1+\delta+\delta^2)u(M) + (\delta^3 + \cdots + \delta^{N+2})u(c^* - \eta) + \frac{\delta^{N+3}u(M)}{1-\delta} \\ &< \frac{c^*}{1-\delta^2} - \delta^{\nu/\epsilon} = v(\mathbf{k}^*) - \delta^{\nu/\epsilon} \leq V(\kappa^*) - \delta^{\nu/\epsilon}. \end{aligned} \tag{19}$$

The first inequality above follows from the fact that consumption in periods 1 and 2 is at most $M$. Because at most one further period of consumption $c^* - \eta$ or better can be sustained over the next $N+1$ periods, I bound total payoff by placing this round in period 3, following it up with $N$ rounds of consumption $c^* - \eta$ or less. Thereafter, I bound consumption by $M$ again. The second inequality follows from (11), and the very last inequality is true by definition. ∎

Claim 6. *Let* $\mathbf{k}$ *be a feasible path from some initial stock in* $[0, K]$, *and suppose that associated consumption on this path satisfies* $c_t \geq c^* - \eta$ *for* $t = 1, 2$. *Then* $v(\mathbf{k}) < V(\kappa^*) - \delta^{\nu/\epsilon}$.

*In particular, no such path can be optimal from* $\kappa^*$.

*Proof.* By Claim 4, $k_2 < \frac{3}{2}$. Now apply Claim 5. ∎

Say that a function $F(x)$ is *locally (strictly) convex* at $x$ if there exists an interval $I$ with $x \in \text{int}I$ such that $F$ is (strictly) convex on $I$.

Claim 7. *For continuous, nondecreasing functions* $u$ *and* $g$, *a discount factor* $\rho \in (0, 1)$, *and constants* $y$ *and* $x$, *consider the two-period maximization problem:*

$$\max_{(c_1, c_2, k) \geq 0} u(c_1) + \rho u(c_2)$$

*subject to*

$$c_1 + k = y \text{ and } c_2 + x = g(k).$$

*Suppose that* (a) $(c_1, c_2, k) \gg 0$; (b) $u$ *is locally strictly convex around* $c_1$ *and* $c_2$, *and* (c) $f$ *is locally convex around* $k$.

[6] Given (8), we have $(1+\alpha)\eta + \alpha^2(1-\beta) + (2-\alpha\beta) < (1+\alpha)\alpha(1-\beta) + \alpha^2(1-\beta) + (2-\alpha\beta) \leq 3(1-\beta) + (2-\alpha\beta) \leq 3/2$, where the last inequality follows from $\alpha, \beta \in (0.9, 1)$.

*Then $(c_1, c_2, k)$ cannot be a solution to the maximization problem.*

*Proof.* Suppose that $(c_1, c_2, k) \gg 0$. There exists $\mu > 0$ such that $u$ is strictly convex on $[c_i - \mu, c_i + \mu]$, $i = 1, 2$, and $f$ is convex on $[k - \mu, k + \mu]$. Define $(c_1', c_2', k') \geq 0$ and $(c_1'', c_2'', k'') \geq 0$ by $k' \equiv k - \mu$, $k \equiv k + \mu$, $c_1' \equiv c_1 + \mu$, $c_1' \equiv c_1 - \mu$, $c_2' = g(k - \mu) - x$, and $c_2'' = g(k + \mu) - x$. Then, using the local convexity of $u$ and $g$ at their respective points,

$$\begin{aligned}
\frac{1}{2}\left[u(c_1') + \rho u(c_2')\right] + \frac{1}{2}\left[u(c_1'') + \rho u(c_2'')\right] &> u\left(\frac{c_1' + c_2'}{2}\right) + \rho u\left(\frac{c_1' + c_2'}{2}\right) \\
&= u(c_1) + \rho u\left(\frac{1}{2}g(k - \mu) + \frac{1}{2}g(k + \mu) - x\right) \\
&\geq u(c_1) + \rho u\left(g(k) - x\right).
\end{aligned}$$

But this means that $(c_1, c_2, k)$ cannot be a solution to the maximization problem (either $(c_1', c_2', k') \geq 0$ or $(c_1'', c_2'', k'')$ will dominate it). ∎

Claim 8. *Suppose that* $\mathbf{k}$ *is a feasible path from* $\kappa > 0$*, such that for two dates* $t$ *and* $s$ *with* $t < s$*, we have* $0 < c_\tau \neq c^*$ *for* $\tau = t, s$*, and* $k_\tau \neq 1, 2, 2 + \beta$ *for all* $\tau = t, \ldots, s - 1$*. Then* $\mathbf{k}$ *cannot be optimal.*

*Proof.* Fix a feasible program $\mathbf{k}$ and let $\{c_\tau\}$ be the associated consumption sequence. For $\tau = t, \ldots, s - 1$, define a function $r_\tau(x)$ by $r_\tau(x) \equiv \max\{f(x) - c_{\tau+1}, 0\}$. Define a function $g$ by

$$g(x) = r_{s-1} \circ \cdots \circ r_t(x) \tag{20}$$

for all $x > 0$.

Suppose, contrary to our assertion, that $\mathbf{k}$ is optimal. Consider the problem

$$\max u(c_t') + \delta^{s-t} u(c_s') \tag{21}$$

over all feasible paths $\mathbf{k}'$ that have associated consumption $c_\tau' = c_\tau$ for $\tau \neq t, s$. If our path $\mathbf{k}$ is to solve this problem (as it surely must if $\mathbf{k}$ is optimal from $\kappa$), then in particular $(c_t', c_s') = (c_t, c_s)$ must solve the maximization problem in Claim 7, with $y = f(k_{t-1})$, $x = k_s$, and $\rho = \delta^{s-t}$.

Note that $g$ is continuous and nondecreasing. Consider all the input values implicitly defined by $g(k_t)$ — using the functions $r_\tau$ — up to date $s - 1$; they coincide exactly with the values along $\mathbf{k}$. For each such date, $f(k_\tau) > c_{\tau+1}$ (because $c_s > 0$). Therefore $r_\tau(x) = f(x) - c_{\tau+1}$ locally around $x = k_\tau$. Because $k_\tau \neq 1, 2, 2 + \beta$, each of these functions is also locally convex around $k_\tau$. So $g$ is locally convex at $k = k_t$.

Because $0 < c_\tau \neq c^*$ for $\tau = t, s$, Claim 1 tells us that $u$ is locally strictly convex at $c_t$ and $c_s$. Invoking Claim 7, we see that (21) cannot be solved by $(c_t, c_s, k_t)$. But this means that $\mathbf{k}$ cannot be optimal, a contradiction. ∎

Claim 9. *Let* $\mathbf{k}$ *be optimal from initial stock* $\kappa^*$*. Then, if* $c_1 \neq c^*$*,* $c_2 = 0$*.*

*Proof.* Suppose that $c_1 \neq c^*$. Because $c_1 \neq c^*$, we know that $k_1 \neq 2$. By Claim 3, $c_1 \geq c^* - \eta$, so $k_1 \leq 2 + \eta < 2 + \beta$, by (8).[7] By Claim 4, $k_1 > 3/2 > 1$. Finally, by Claim 6, we know that $c_2 \leq c^* - \eta$.

Therefore, if $c_2 > 0$, we know that $c_2 \in (0, c^*)$. Now $(c_1, c_2, k_1)$ satisfies all the assumptions of Claim 8, so that **k** cannot be optimal, a contradiction. ∎

In what follows, remember that $\sigma(k) \equiv f(f(k)) - k$ for $k \in [0, K]$, and that $\xi(\epsilon)$ is defined by $f(\xi(\epsilon)) \equiv 2$.

Claim **10.** *For every* $k < 2$,

$$\sigma(2) - \sigma(k) \geq \min\left\{\left(\frac{\alpha\beta}{\epsilon} - 1\right)(2-k), (\alpha+1)\beta - S\right\} > 0, \tag{22}$$

*where* $S$ *is the maximum value of* $\sigma(k)$ *on* $[0, \xi(\epsilon)]$.

*Proof.* Recall from Claim 2 that $\sigma(2) > \sigma(k)$ for all $k \neq 2$. In particular,

$$\sigma(2) - \sigma(k) \geq (\alpha+1)\beta - S > 0 \tag{23}$$

for all $k \in [0, \xi(\epsilon)]$. When $k \in (\xi(\epsilon), 2)$, we know from the definition of $\xi(\epsilon)$ that $f(k) > 2$, so that

$$\sigma(k) = \alpha\left[\beta - \frac{\beta}{\epsilon}(2-k)\right] + (2+\beta) - k.$$

It follows that

$$\sigma(2) - \sigma(k) = (\alpha+1)\beta - \alpha\left[\beta - \frac{\beta}{\epsilon}(2-k)\right] - (2+\beta) + k = \left(\frac{\alpha\beta}{\epsilon} - 1\right)(2-k). \tag{24}$$

Observe that $\alpha\beta/\epsilon > 1$, because $(\alpha, \beta) \geq (0.9, 0.9)$ and $\epsilon < 1 - \beta$.

Combining (23) and (24), we must conclude that (22) holds. ∎

Claim **11.** *Suppose that* $0 < \kappa = \kappa^* - \Delta$, *for some* $\Delta > 0$. *Then associated consumption along some feasible path* **k** *from* $\kappa$ *can coincide with the sequence* $(c^*, 0, c^*, 0, c^*, \ldots)$, *starting from date 1, for at most* $T(\Delta)$ *consecutive periods, where*

$$T(\Delta) \leq R(\Delta, \epsilon) \equiv 4\max\left\{\frac{\epsilon}{(\alpha\beta - \epsilon)\Delta}, \frac{1}{(\alpha+1)\beta - S}\right\} + 1. \tag{25}$$

*Proof.* Consider associated consumption along some feasible path **k** from $\kappa$, where $0 < \kappa = \kappa^* - \Delta$. Suppose that starting from date 1, it coincides with $(c^*, 0, c^*, 0, c^*, \ldots)$ for $T$ consecutive periods (at this stage $T$ may be infinite). Then for every odd $t \leq T - 2$, it is easy to see that $k_t < 2$, and invoking Claim 10,

$$\begin{aligned} k_{t+2} &\leq k_t - [\sigma(2) - \sigma(k_t)] \\ &\leq k_t - \min\left\{\left(\frac{\alpha\beta}{\epsilon} - 1\right)(2-k_t), (\alpha+1)\beta - S\right\}, \end{aligned}$$

[7] (8) informs us that $\eta < \alpha(1-\beta) < 0.1$, while $\beta > 0.9$.

where, as before, $S$ is the maximum value of $\sigma(k)$ on $[0, \xi(\epsilon)]$. By (22), $k_{t+2} < k_t$ for all odd $t \leq T-2$, so that

$$k_{t+2} \leq k_t - \min\left\{\left(\frac{\alpha\beta}{\epsilon} - 1\right)(2-k_1), (\alpha+1)\beta - S\right\}.$$

However, as long as $T \geq 2$, we know that $2 - k_1 = \kappa^* - \kappa = \Delta$, so that for every odd $3 \leq t \leq T$,

$$0 \leq k_t \leq 2 - \frac{t-1}{2}\min\left\{\left(\frac{\alpha\beta}{\epsilon} - 1\right)\Delta, (\alpha+1)\beta - S\right\}.$$

Consequently

$$T(\Delta) \leq 4\max\left\{\frac{\epsilon}{(\alpha\beta-\epsilon)\Delta}, \frac{1}{(\alpha+1)\beta - S}\right\} + 1.$$

∎

Claim **12.** *Let* $\mathbf{k}$ *be optimal from* $\kappa^*$*. Then* $c_1 \geq c^*$*.*

*Proof.* Suppose not. Then (using Claim 3) we know that $0 < c_1 < c^*$. By Claim 9, $c_2 = 0$. I now assert that:

(26) $$c_t = c^* \text{ when } t \text{ is odd, and } c_t = 0 \text{ when } t \text{ is even.}$$

for all $t \geq 3$. Suppose this is false; consider the first odd date $t \geq 3$ for which either $c_t \neq c^*$ or $c_{t+1} \neq 0$. Consider the continuation program from date $t$ with initial stock $k_{t-1}$. Because $c_1 < c^*$ and moreover, $(c_s, c_{s+1}) = (c^*, 0)$ for all $3 \leq s < t$ (with $s$ odd), it must be the case that $k_{t-1} > \kappa^*$. Moreover, the continuation path of $\mathbf{k}$ is optimal from $k_{t-1}$. It follows from Claims 3 and 6 that $c_t \geq c^* - \eta$ and $c_{t+1} < c^* - \eta$.

Therefore if $c_t \neq c^*$, $c_t$ must be strictly positive, while if $c_{t+1} > 0$, it must be that $c_{t+1} \neq c^*$. Thus, in either case, at the first date $s$ in which our assertion fails ($t$ or $t+1$), $0 < c_s \neq c^*$.

It is easy to check that for all $1 \leq \tau \geq s$, $2 + \beta > k_\tau > 2$ for $\tau$ odd and $k_\tau > 2 + \beta$ for $\tau$ even. Therefore all the conditions of Claim 8 are satisfied, so that $\mathbf{k}$ must be suboptimal, a contradiction. Therefore (26) is true.

We have therefore shown that the consumption sequence associated with $\mathbf{k}$ satisfies $c_1 < c_1^*$, and $c_t = c_t^*$ for all $t \geq 2$, where $\{c_t^*\}$ is the consumption sequence associated with $\mathbf{k}^*$. Once again, this contradicts the optimality of $\mathbf{k}$, simply because $\mathbf{k}^*$ is feasible from $\kappa^*$. ∎

We now complete the proof of the proposition. It will suffice to show that if $\mathbf{k}$ is an optimal program from $\kappa^*$, then

(27) $$c_1 = c^* \text{ and } c_2 = 0.$$

By Claim 12, we know that $c_1 \geq c^*$. Moreover, Claim 9 tells us that if $c_1 > c^*$, then $c_2 = 0$. Therefore, if (27) is false, we have

(28) $$c_1 \geq c^* \text{ and } c_2 \geq 0, \text{ with } \textit{exactly} \text{ one strict inequality.}$$

Let $\Delta \equiv \max\{c_1 - c^*, c_2\}$. In what follows, we explore the properties of the continuation path from $\kappa = k_2$. By (28), we have $\kappa^* - \kappa = \Delta > 0$.

By Claim 11, we know that the continuation consumption sequence (starting from date 2) can coincide with $(c^*, 0, c^*, 0, \dots)$ for at most $T(\Delta)$ periods, where $T(\Delta)$ satisfies (25). Let $\tau$ be the very first date (starting from 3) at which the coincidence ends. Then

$$\tau \le T(\Delta) + 2 \le R(\Delta, \epsilon) + 3. \tag{29}$$

We first consider two cases (they do not exhaust all the possibilities).

*Case 1.* $c_{\tau-1} = 0$ and $c_\tau < c^* - \eta$. Consider the continuation program from $k_{\tau-1}$ and apply Claim 3. It is immediate that

$$V_\tau \le V(\kappa^*) - \delta^{\nu/\epsilon}. \tag{30}$$

*Case 2.* $c_{\tau-1} = c^*$ and $c_\tau \ge c^* - \eta$. Consider the continuation program from $k_{\tau-2}$ and apply Claim 6. It is immediate that

$$V_{\tau-1} \le V(\kappa^*) - \delta^{\nu/\epsilon}. \tag{31}$$

We will deal with both these cases together. Let $T$ be a time index that stands for either $\tau$ or $\tau - 1$, depending on whether we are in Case 1 or Case 2.

Consider an alternative path $\mathbf{k}'$ in which $c_1'$ is reduced from $c_1$ to $c^*$, $c_2'$ is reduced from $c_2$ to 0 (only one strict reduction is involved, by (28)) and which coincides with $\{c^*, 0, c^*, 0, \dots\}$ up to and including date $T - 1$. Notice that $k'_{T-1} = \kappa^*$. From date $T$ onwards, let $\mathbf{k}'$ coincide with an optimal path from $\kappa^*$.

Discounting all payoffs to period 1, and letting $L$ stand for the utility loss in periods 1 or 2 (in moving from $\mathbf{k}$ to $\mathbf{k}'$), it is easy to see that

$$\begin{aligned} v(\mathbf{k}') - v(\mathbf{k}) &= \delta^{T-1}\left[V(\kappa^*) - V_T\right] - L \\ &\ge \delta^{T-1}\delta^{\nu/\epsilon} - L \\ &\ge \delta^{R(\Delta,\epsilon)+2}\delta^{\nu/\epsilon} - L, \end{aligned} \tag{32}$$

where the first of the inequalities uses (30) and (31), and the second employs (29).

Now we estimate $L$. If the reduction occurs in period 1,

$$L = \delta^{(M - c^* - \Delta)/\epsilon\Delta} \le \delta^{\alpha(1-\beta)/\epsilon\Delta} \le \delta^{\eta/\epsilon\Delta},$$

where the first inequality follows from the fact that $\Delta \le 2$,[8] and $M = \alpha + \beta + 2$ (see (18)), and the second inequality follows from (8).

If the reduction occurs in period 2,

$$L = \delta^{(c^* - \Delta)/\epsilon\Delta} \le \delta^{\eta/\epsilon\Delta},$$

where the inequality follows from the fact that $c_2 \le c^* - \eta$ (by Claims 3 and 6), so that $c^* - \Delta = c^* - c_2 \ge \eta$. Therefore in either case,

$$L \le \delta^{\eta/\epsilon\Delta}, \tag{33}$$

[8] Available output is $(\alpha + 1)\beta + 2$, while $c^* = (\alpha + 1)\beta$.

where

$$0 < \Delta \le 2 \tag{34}$$

(The last inequality follows from the fact that $c^* - \eta = (\alpha+1)\beta - \eta < 2$.)

Combining (32) and (33), we see that

$$v(\mathbf{k}') - v(\mathbf{k}) \ge \delta^{R(\Delta,\epsilon)+2}\delta^{\nu/\epsilon} - \delta^{\eta/\epsilon\Delta}. \tag{35}$$

I claim that for all $\Delta \in (0,2]$,

$$\frac{\nu}{\epsilon} + R(\Delta,\epsilon) + 2 < \frac{\eta}{\epsilon\Delta}. \tag{36}$$

We recall the definition of $R(\Delta,\epsilon)$ from (25) and accordingly break up the proof of (36) into two steps. First suppose that

$$R(\Delta,\epsilon) = \frac{4\epsilon}{(\alpha\beta-\epsilon)\Delta} + 1.$$

Then, after slight manipulation, we see that (36) is true if

$$\nu + 3\epsilon < \frac{1}{\Delta}\left[\eta - \frac{4\epsilon^2}{\alpha\beta-\epsilon}\right]. \tag{37}$$

(9) and (15) tell us that the right-hand side of (37) is certainly positive. So, invoking (34), a sufficient condition for (37) to hold is

$$\nu + 3\epsilon < \frac{1}{2}\left[\eta - \frac{4\epsilon^2}{\alpha\beta-\epsilon}\right].$$

Rearranging terms and using (9), this is equivalent to the inequality

$$3\epsilon + \frac{2\epsilon^2}{\alpha\beta-\epsilon} < \frac{\eta}{2} - \nu = \nu,$$

which is guaranteed by (15).

Second, suppose that

$$R(\Delta,\epsilon) = \frac{4}{(\alpha+1)\beta - S} + 1.$$

Then (36) is once again true if

$$\frac{\nu}{\epsilon} + \frac{4}{(\alpha+1)\beta - S} + 3 < \frac{\eta}{\epsilon\Delta}. \tag{38}$$

To establish (38), it is sufficient to verify that

$$\epsilon\left[\frac{4}{(\alpha+1)\beta - S} + 3\right] < \frac{\eta}{c^* - \eta} - \nu = \nu,$$

where the equality invokes (9). But this is guaranteed by (17).

This establishes (36). However, combining (35) and (36), we are forced to conclude that $v(\mathbf{k}') > v(\mathbf{k})$, but this contradicts the presumed optimality of $\mathbf{k}$. Therefore (27) must indeed be true in both Cases 1 and 2.

That leaves only

*Case 3.* $0 < c_T \neq c^*$, with the additional proviso that

$$c_T \geq c^* - \eta \text{ if } c_{T-1} = 0. \tag{39}$$

Recall that (28) holds. Therefore we have either $c_1 > c^*$ or $c_2 > 0$ (but not both). We also know from Claims 3 and 6 that $c_2 \leq c^* - \eta$. Therefore, we have a date $t$ (equal to 1 or 2) and another date $s$ (equal to $T > 2$) at which $0 < c_\tau \neq c^*$. Moreover, for any $\tau$ strictly between these two dates the consumption path has $c_\tau = c^*$ for $\tau$ odd, and $c_\tau = 0$ for $\tau$ even.

Now we make the following observations about the path of capital stocks:

(a) At no date $t$ between 1 and $T$ can $k_t$ be less than $3/2$. For if this were true, then by Claim 5, we see that (30) must hold for some $\tau \leq T(\Delta)+2$, and exactly the same argument for Cases 1 and 2 applies to obtain a contradiction. In particular, $k_t \neq 1$ for all such dates.

(b) For $\tau$ odd, we have $k_\tau < 2$.

(c) For $\tau$ even, we have $k_\tau < 2+\beta$. Moreover, for $\tau$ even, $k_\tau \neq 2$. To see this, note that because $\tau$ is even, $c_\tau = 0$. By (39), we see that $c_{\tau+1} \geq c^* - \eta$ whether $T > \tau + 1$ or $T = \tau + 1$. Therefore $k_{\tau+1} \leq f(k_\tau) - c^* + \eta = 2 + \beta - (\alpha+1)\beta + \eta < 3/2$, using (8.[9]). But this contradicts part (a). So $k_\tau \neq 2$ for $\tau$ even.

Combining (a)–(c), we see that for all $\tau = t, \ldots, s-1$, $k_\tau \neq 1, 2, 2+\beta$. Now all the conditions of Claim 8 are satisfied, which means that $\mathbf{k}$ cannot be optimal. This contradiction completes the proof of the proposition. ∎

## References


[1] T. Mitra and D. Ray (1984), Dynamic Optimization on a Non-Convex Feasible Set: Some General Results for Non-Smooth Technologies, *Zeitschrift fur Nationalokonomie* **44**, 151–175.
[2] T. C. Koopmans (1960), Stationary Ordinal Utility and Impatience, *Econometrica* **28**, 287–309
[3] E. S. Phelps (1962), The End of the Golden Age in Solovia: Comment, *American Economic Review* **52**, 1097–1099.
[4] E. S. Phelps (1965), Second Essay on the Golden Rule of Accumulation, *American Economic Review* **55**, 793–814.
[5] F. P. Ramsey (1928), A Mathematical Theory of Saving, *Economic Journal* **38**, 543–559.


[9] By (8), $2 + \beta - (\alpha+1)\beta + \eta < 2 + \beta - (\alpha+1)\beta + \alpha(1-\beta) = 2 + \alpha - 2\alpha\beta < 3/2$, because $\alpha, \beta \in (0.9, 1)$.